# Polynomial Functions

A polynomial function is a sum of multiples of an independent variable raised to various integer powers. The general form of a polynomial function is

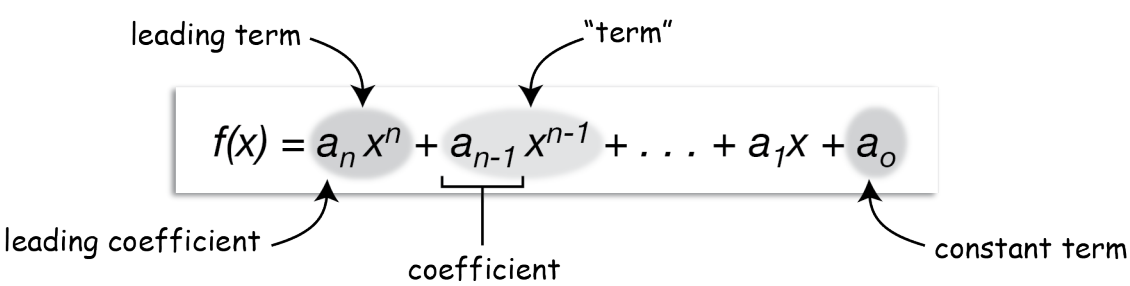
$$f(x) = a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n x^0 \quad ,$$

where the $\{ a_n \} = \{ a_o, a_1, a_2, \ldots, a_n \}$ are constants, and $x$ is the independent variable. The **term** containing $x^n$, the highest power of $x$, is called the **leading term**, and $n$ is called the **degree** of the polynomial. Some common named polynomial functions have the forms:

| | |
|---|---|
| $f(x) = Ax + B$ | **Linear** polynomial function—highest power of x is 1 |
| $f(x) = Ax^2 + Bx + C$ | **Quadratic**—highest power of $x$ is 2 |
| $f(x) = Ax^3 + Bx^2 + Cx + D$ | **Cubic**—highest power of $x$ is 3 |
| $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$ | **Quartic**—highest power of $x$ is 4, where $A$, $B$, $C$, $D$ and $E$ are constants. |

The graphs of polynomial functions are always smooth curves with no discontinuities like asymptotes, holes or sharp turns. Polynomial functions are used to model a wide variety of natural phenomena, and natural and engineered shapes. Polynomial functions will also turn out to be the *solutions* of many equations you'll solve in the future—*differential equations* you'll encounter after calculus.

**Anatomy of a polynomial function**




The polynomial function $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$, for example, is made of 5 individual **terms**. $Ax^4$ is the **quartic term**, $Bx^3$ is the **cubic term**, $Cx^2$ is the **quadratic term**, $Dx$ is the **linear term** and $E$ is the **constant term**. The highest exponent present in the polynomial is its **degree**. A cubic polynomial, for example, is a "**third-degree polynomial**," or a "**polynomial of degree 3**"

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## Zeros or Roots of a polynomial function

We are often interested in the **zeros** of a polynomial function, the values of $x$ that solve

$$a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n x^0 = 0 \quad .$$

These are the **x-intercepts** of the polynomial graph, and they may or may not be purely real numbers (remember that all numbers are complex numbers, some have no imaginary parts and are therefore purely real).

Note that the solution to any polynomial equation can be turned into a hunt for roots. For example, the equation $5x^4 - 3x^3 - 2x + 4 = 27$ can be written as $5x^4 - 3x^3 - 2x + -23 = 0$ by shifting the constant to the left side.

The **fundamental theorem of algebra** says that every polynomial function of degree $n$ has exactly $n$ complex roots. Here we note that real numbers are complex numbers without an imaginary part. There are several techniques for finding the roots of polynomial functions. Below are five methods for getting to the roots of polynomials.

## Methods of finding roots

### I. Finding the greatest common factor (GCF)

Identify the greatest common factor of every term of the polynomial. If you can find a GCF, this is always the **first and easiest step** in factoring a polynomial function.

**Example**: $f(x) = 14x^5 - 4x^3 + 2x$

$f(x) = 2x(7x^4 - 2x^2 + 1)$ — $2x$ is a common factor of all terms.

This function needs a little more work before we find the roots (it has only one real root), but you get the idea.

**Example**: $f(x) = x(3x + 1) + 5(3x + 1)$

$f(x) = (3x + 1)(x + 5)$ — The binomial $(3x+1)$ is a common factor.

### II. Grouping

Sometimes it's possible to find different GCFs for different parts of the polynomial by grouping terms in different ways.

**Example**: $f(x) = x^3 + 3x^2 + 2x + 6$

$f(x) = (x^3 + 3x^2) + (2x+6)$ — Group the polynomial like this.

$f(x) = x^2(x + 3) + 2(x+3)$ — Find the GCF of each grouping.

$f(x) = (x + 3)(x^2 + 2)$ — Now pull out the GCF $(x+3)$
… and he roots are easy to find.

**Example**: $f(x) = 7x^3 - 14x^2 - x + 2$

$f(x) = (7x^3 - 14x^2) - (x - 2)$ Group

$f(x) = 7x^2(x - 2) - (x - 2)$ Find the GCF of each grouping

$f(x) = (x - 2)(7x^2 - 1)$ Now pull out the GCF $(x-2)$.

## III. Recognizing the form of a quadratic

Sometimes a polynomial can resemble a quadratic equation enough that substitution of variables can help to turn it into one so that it can be solved in two steps.

**Example**: $f(x) = x^4 + 2x^2 - 8$ Let $y = x^2$, then

$f(x) = y^2 + 2y - 8$ which can be factored ...

$f(x) = (y + 4)(y - 2)$ The roots of $f(y)$ are $y = -4, 2$

continued ...

Solve for x: $x^2 = -4 \Rightarrow x = \pm 2i$

$x^2 = 2 \Rightarrow x = \pm\sqrt{2}$ The roots of $f(x)$ are $\pm 2i$, $\pm\sqrt{2}$

## IV. Sums or Difference of Cubes

Sometimes you will encounter polynomials that are sums or differences of cubic terms like $(x^3 - 8)$ or $(27y^3 + 64)$. Both equations can be rewritten like this:

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ or

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ ← You should verify these formulae for yourself

**Example**: $f(x) = 27x^3 - 8$ Recognize $27x^3$ and $8$ as the cubes of $3x$ and $2$.

$f(x) = (3x - 2)(9x^2 - 6x + 4)$

**Example**: $f(x) = 64x^3 + 1$ Recognize $64x^3$ and $1$ as the cubes of $4x$ and $1$.

$f(x) = (8x + 1)(16x^2 - 4x + 1)$

There is no need to memorize these sum and difference formulae; they can be looked up when you need them. But it would be wise to understand how they are derived from first principles.

## V. The Rational Root Theorem

Let $f(x)$ be a polynomial of degree $n$ with leading coefficient $a_n$ and constant term $a_0$. *If* the function has any rational roots at all (and it might not), they have the form $x = p/q$, nonzero rational numbers, where $p$ must be a factor of $a_0$ and $q$ must be a factor of $a_n$.

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**Example**: $f(x) = 3x^4 + 4x^3 + x + 2$ *p* are factors of $a_0=2$ & *q* are factors of $a_4=3$

$p = \pm 1, \pm 2$ & $q = \pm 1, \pm 2, \pm 3$ so $\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$

Now we just test whether the p/q are roots by synthetic substitution:

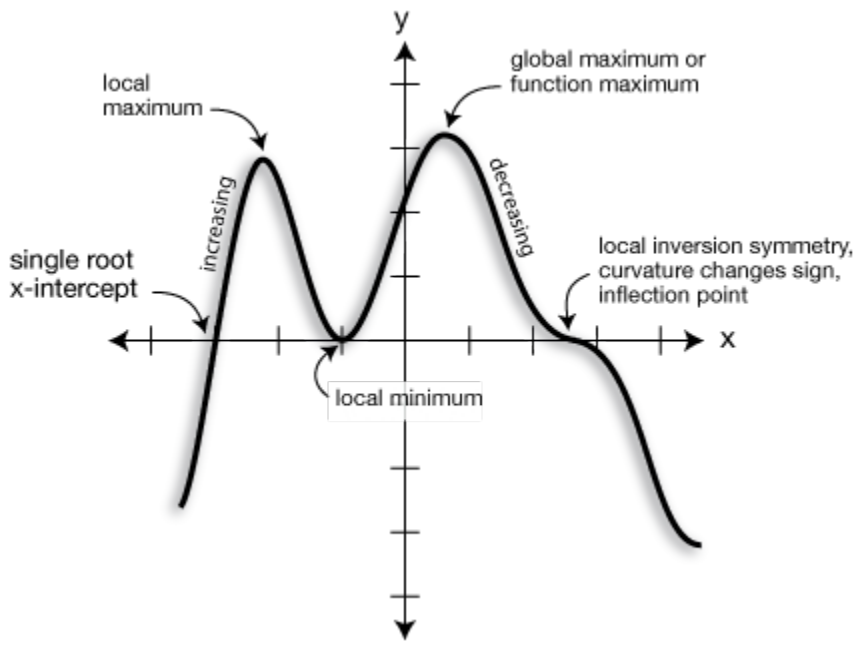
```
1    3  4  0  1  2      -1    3  4  0  1  2      2    3  4  0  1  2
        3  7  7  8            -3 -1  1 -2            6 16 32-66
     3  7  7  8 10         3  1 -1  2  0         3  8 16 33-64
```

We could have stopped at x=-1, which has no remainder, so it's a root. There's no need to test the rest of the candidates right now.

$f(x) = (x + 1)(3x^3 + x^2 - x + 2)$ Now the cubic polynomial can be factored the same way or by one of the previous methods if one works.

## Polynomial graphs

Graphs of polynomial functions are always smooth curves. They can include local or global maxima or minima or inflection points, as shown on this example graph.




**local maximum**
*a point higher than neighboring points on either side*
**local minimum**
*a point lower than neighboring points on either side*
**global maximum**
*the highest point on the graph*
**global minimum**
*the lowest point on the graph*

As you study the graphs below, note that these maxima or minima may or may not exist for a given polynomial function.

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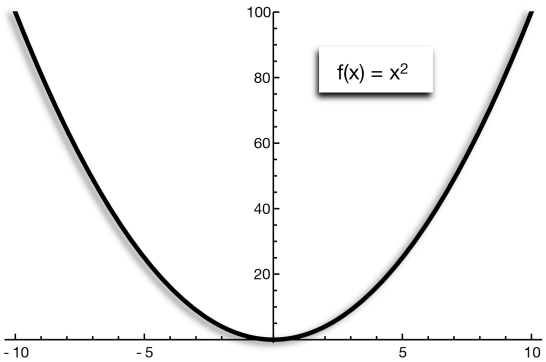
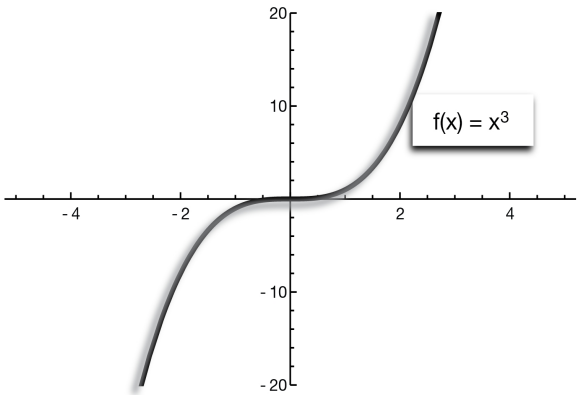
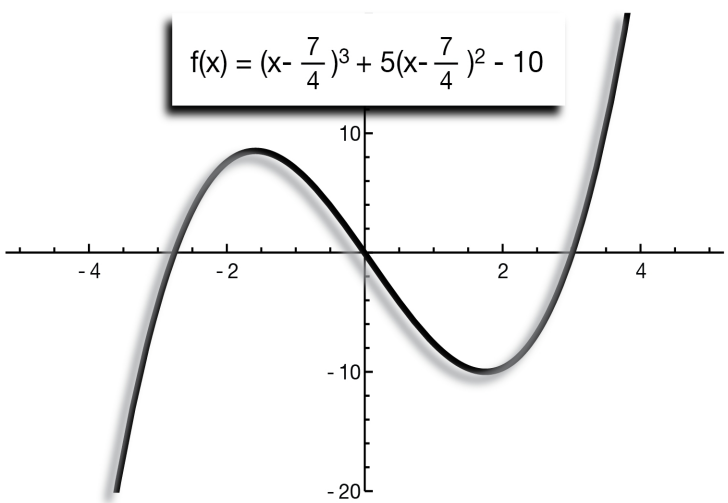
The graph of any **quadratic function** is a **parabola** (right), one of the **conic section**s. A parabola always has either a **global maximum** or **global minimum**, a point that is higher or lower, respectively, than every other point on the graph. A parabola has exactly one **turning point**, a point where the slope of the curve changes from positive to negative, or (–) → (+). Note the end behavior of a parabola: both ends grow without bound in the same direction—in either the positive or negative y-direction. This is true of all functions of even degree—quartic, sextic, &c.




The graph of a cubic function can have an inflection point—a point at which the curvature changes sign. In the graph of $f(x) = x^3$, the inflection point is at the origin. The ends of a cubic function grow without bound in opposite directions. Cubic functions can have two turning points, as shown in the graph of

$$f(x) = \left(x - \frac{7}{4}\right)^3 + 5\left(x - \frac{7}{4}\right)^2 - 10$$




Note that the graph of $f(x) = x^3$ (above) has one root (actually it's a **triple root**), while the graph at left has three real roots (three x-intercepts). Cubic functions either have one real root and two complex roots, or all real roots. Furthermore, the complex roots are always complex conjugates. Cubic functions can also have local maxima or minima. This function has one of each.

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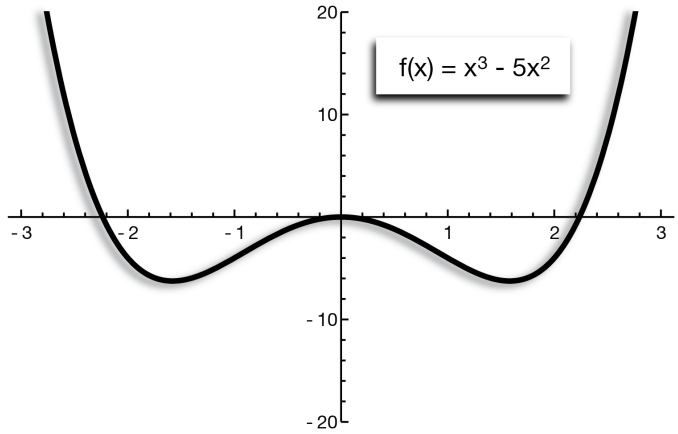
The graphs of polynomial functions of higher degree can have more x-intercepts, more turning points and more local maxima or minima. The quartic function plotted here has two equal global minima, a local maximum, four real roots (one of them a **double root** at the origin).




**General features of polynomial graphs**

- For a polynomial of degree *n*, there are (at most) *n-1* turning points. For example, a cubic polynomial (degree *3*) has no more than two turning points (see our two examples above). At a turning point, the slope of the curve changes from negative to positive or from positive to negative—the slope changes sign.
- In general, the graphs of cubic polynomials look like sideways **S**-curves of various shapes. Graphs of quartic functions look like **W**s or **M**s.
- X-intercepts (roots) can either cross the axis (**multiplicity** of 1), just touch the axis (multiplicity of 2, or a **double root**—just like a parabola that has its vertex on the x-axis) or be inflection points, where the *curvature* of the graph changes sign (multiplicity of 3, or a **triple root**).
- The behavior of the ends of a polynomial graph, where $x \to \pm\infty$, is determined by the sign of the leading coefficient (see box below).

**<u>Sketching polynomial graphs</u>**

You will need to know how to make quick **sketches** of the graphs of polynomial functions. It's not too difficult if you can figure out a few key things about the function:

1. Determine all of the roots. Find all of the *x*-intercepts and determine whether the graph crosses the axis (single root), just touches it (double root), or whether it is an inflection point (triple root).
2. Determine the *y*-intercept: *(0, f(0))*.
3. Use the sign of the leading coefficient (see below) to determine the behavior of the ends of the graph.
4. Plot a few more points, at least one between each root to see whether the graph is positive or negative there.

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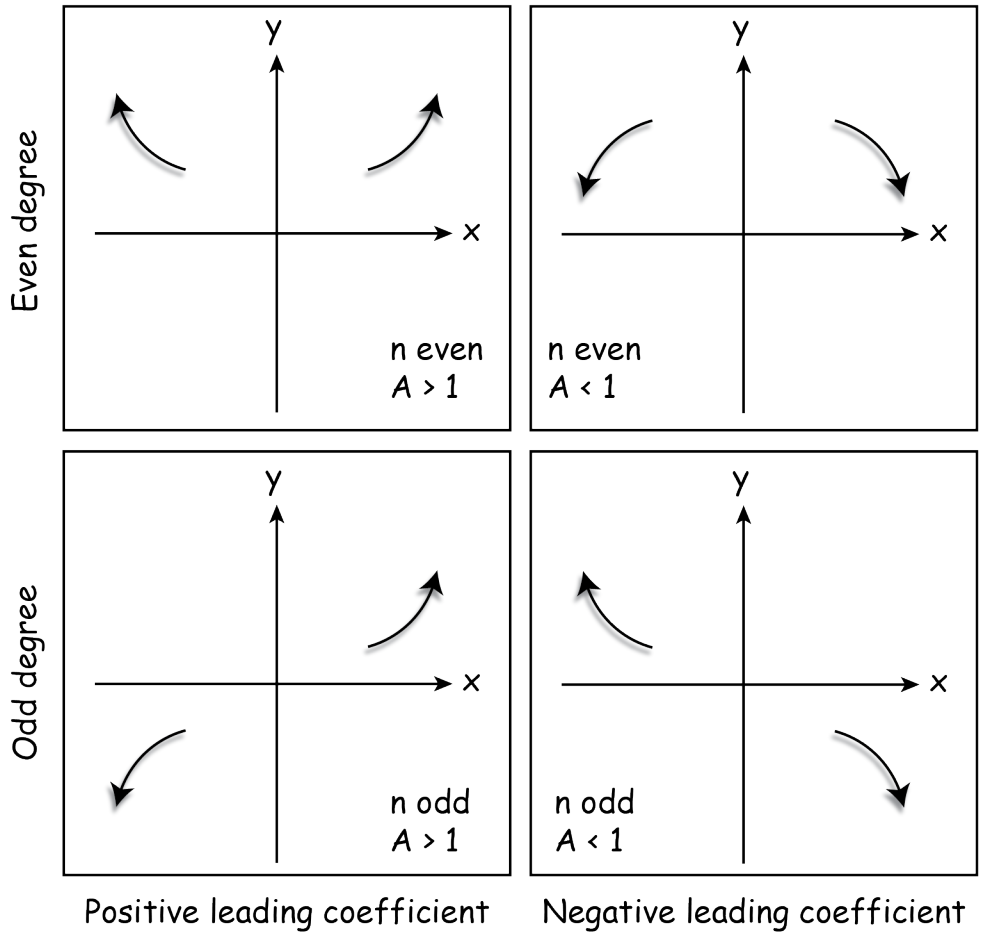
**Leading coefficient test:**
If the coefficient of the highest-degree term is $A$, and $n$ is the degree then:

| | | |
|---|---|---|
| $A > 0$ | n is even | the graph increases without bound upward at both ends |
| $A < 0$ | n is even | the graph decreases without bound downward at both ends |
| $A > 0$ | n is odd | the graph increases on the right end and decreases on the left |
| $A < 0$ | n is odd | the graph increases on the left end and decreases on the right |

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**Cube roots**

We are now equipped to explore an interesting problem—*how many cube roots does a real number have*?

For example, to solve the problem $x = \sqrt[3]{8}$, we're really solving the problem $x^3 = 8$, or better yet, $x^3 - 8 = 0$. This is just a simple cubic polynomial equation, which, according to the fundamental theorem of algebra, must have three solutions. We already know that 2 is a solution, but what about the other two?

Recall from above that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ ← *formula for the difference of cubes*

We can use this to solve $x^3 - 8 = 0$, which is a difference of perfect cubes:

$$x^3 - 8 = (x-2)(x^2+2x+4) \quad \leftarrow \text{2 is easily recognized as a root}$$

$$x^2+2x+4 = 0$$

$$x^2+2x+1 = -3$$

$$(x+1)^2 = -3$$

$$x = -1 \pm i\sqrt{3}$$

To confirm that these are cube roots, just cube one:

$$(-1+i\sqrt{3})(-1+i\sqrt{3})(-1+i\sqrt{3}) = (-2-2i\sqrt{3})(-1+i\sqrt{3}) = 8$$

So the real number 2 has three cube-roots: x ~ = ~ $2, -1 \pm i\sqrt{3}$

**Note**: Complex roots with nonzero imaginary parts *always* come in complex-conjugate pairs. That means that a real number will have one real 5th root and four other complex roots in two pairs of complex conjugates. The 6th roots of a real number will have two real roots (a ± pair because the negative solution results in a positive number when raised to an even power), and four other solutions in complex-conjugate pairs, and so on …

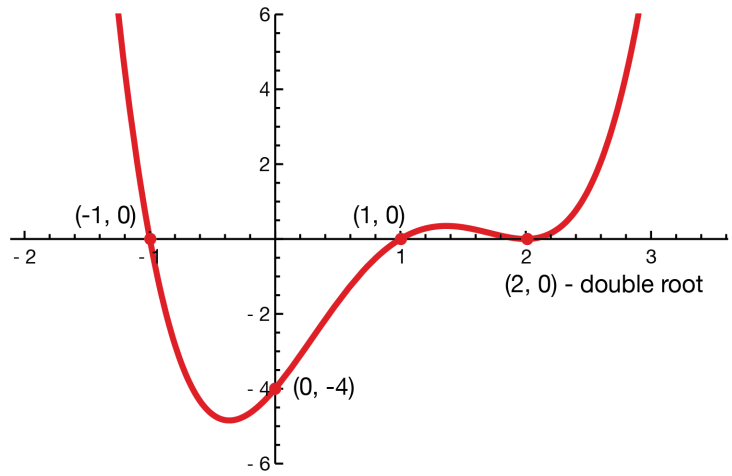
## **Challenge**:

Can you find a *general* formula for the cube roots of a real number? That is, can you find the solutions to the equation $x^3 \pm a^3 = 0$ ?

## Finding equations of polynomial functions from their graphs

In general, a polynomial function of degree $n$ has $n+1$ coefficients that must be found in order to determine its equation uniquely. It is therefore possible to uniquely determine the equation of a polynomial function by knowing $n+1$ points of the function.

For example, if, for a given quartic function, four roots and the y-intercept are known (see graph below), the equation of the function can be determined unambiguously. Here's an example:




By inspection of the graph, we see that the roots of the function are $x = \pm 1$ and 2, where 2 is a double root. That means that the form of the function is

$$f(x) = A(x-1)(x+1)(x-2)^2 \quad ,$$

where the constant $A$ can be determined by using the fifth piece of information available to us, the y-intercept. To find $A$, use the point (0, -4) to write:

$$-4 = A(0-1)(0+1)(0-2)^2 \quad \rightarrow \quad A = 1 \quad .$$

Then $f(x) = (x-1)(x+1)(x-2)^2$

$$f(x) = (x^2-1)(x^2-4x+4)$$

$$f(x) = x^4-4x^3+3x^2+4x-4$$