Rational Functions

In mathematics, **rational** means "in a ratio." A rational function is a ratio of two polynomials. Rational functions have the general form

$$f(x) = \frac{p(x)}{q(x)} ,$$

where *p(x)* and *q(x)* are polynomials, and $q(x) \neq 0$. Here are some examples:

$$f(x) = \frac{x+1}{x-3} \qquad g(x) = \frac{x-2}{x^2+3}$$

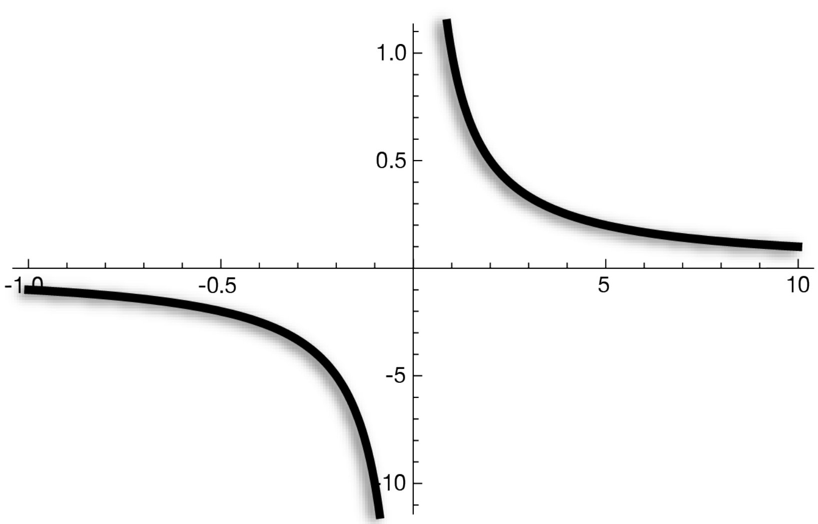
$$h(x) = \frac{x^2-x+7}{2x^3-x^2+3x-1} \qquad k(x) = \frac{x^3}{(x-4)(x+3)}$$

Rational functions are used to model all kinds of important systems. They have uses in chemistry, physics, engineering, economics and business.

The graphs of rational functions are interesting because the denominator can have zeros, and because of the relationship between numerator and denominator as the independent variable approaches infinitely large values (negative or positive). If the denominator of a function becomes zero, the value of the function approaches positive or negative infinity. Here's the simplest possible example:

Parent function of all rational functions: $f(x) = \frac{1}{x}$

We note that for this function x cannot equal zero.



The graph of $f(x) = \frac{1}{x}$ has strange behavior compared to the graphs of polynomial functions because its denominator is zero when *x* = 0.

The graph is discontinuous at x=0. Its value approaches infinity (∞) as x approaches zero from the positive side and –∞ as x approaches zero from the the negative side. It also approaches zero as x grows to ± ∞.

Asymptotes

The graphs of rational functions are characterized by **asymptotes.** An asymptote is a line (or curve) that the graph of a function approaches as the independent variable changes toward some value (the asymptote), but is never actually intersected. The asymptote is not a part of the graph of a function. The graphs of rational functions can display **vertical**, **horizontal**, **slant** and even **polynomial** (curved) asymptotes. We'll mostly be concerned with horizontal and vertical asymptotes.

Vertical Asymptotes

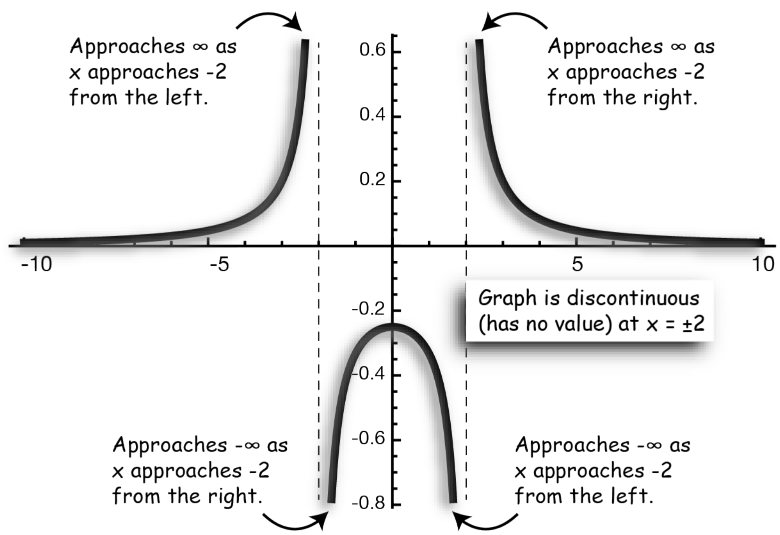
To find vertical asymptotes, we need to find values of the independent variable that make the denominator of a rational function zero. In other words, we need to find the zeros (roots) of the denominator.

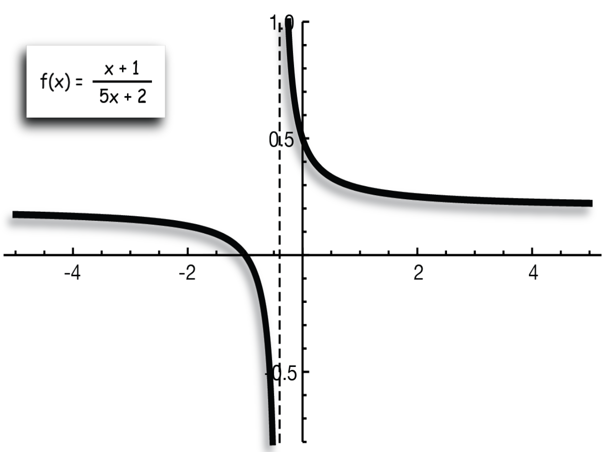
Finding vertical asymptotes: Find the zeros (roots) of the denominator

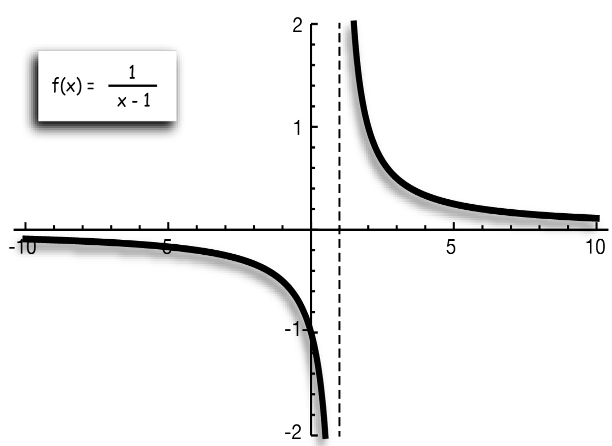
Here are some examples:

$$f(x) = \frac{1}{x^2-4}$$

The denominator has zeros of x= ±2, so the graph of this function will have asymptotes at x = ±2, as shown. For example, as x approaches -2 from the left, the function grows without bound (toward ∞). The function has no value at x= ±2. Two more examples are shown below. Study these and see if you can figure out how vertical asymptotes show up in the graphs of functions.





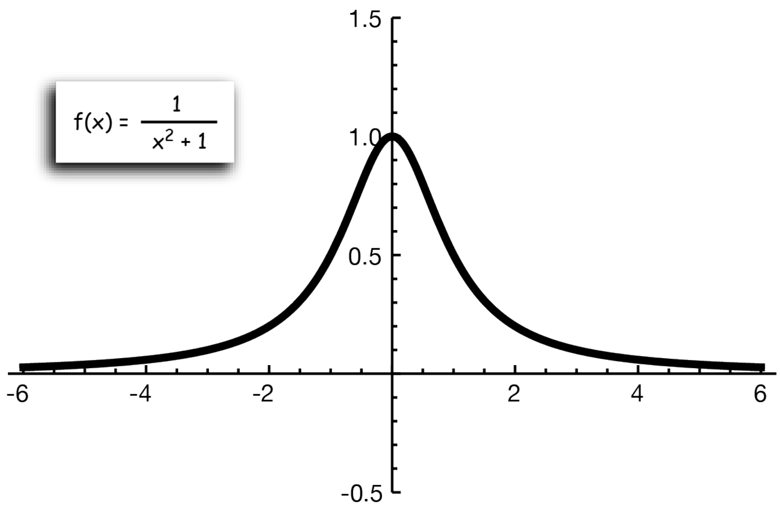


The function plotted at right,

$$f(x) = \frac{1}{x^2+1}$$, has no vertical

asymptotes because the denominator has no real roots. Its roots are $\pm\, i$. It has a horizontal asymptote at $y = 0$.

You will need to watch for non-real zeros of the denominator when you analyze the graphs of rational functions.

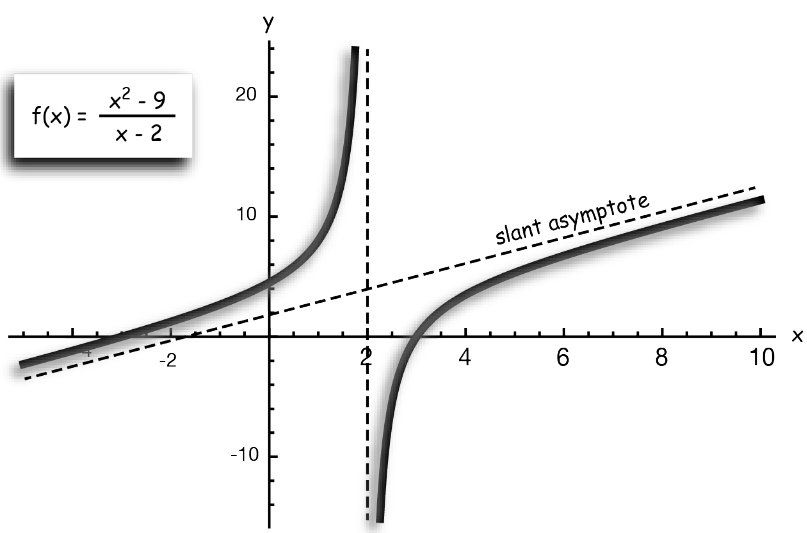


Rational functions with no real zeros of the denominator have no vertical asymptotes.

The same is true for rational function denominators with degree > 2. For example, a rational function with a cubic denominator may only have one vertical asymptote if it has two complex roots. Note, however that such a function *must* have at least one vertical asymptote—*why*?

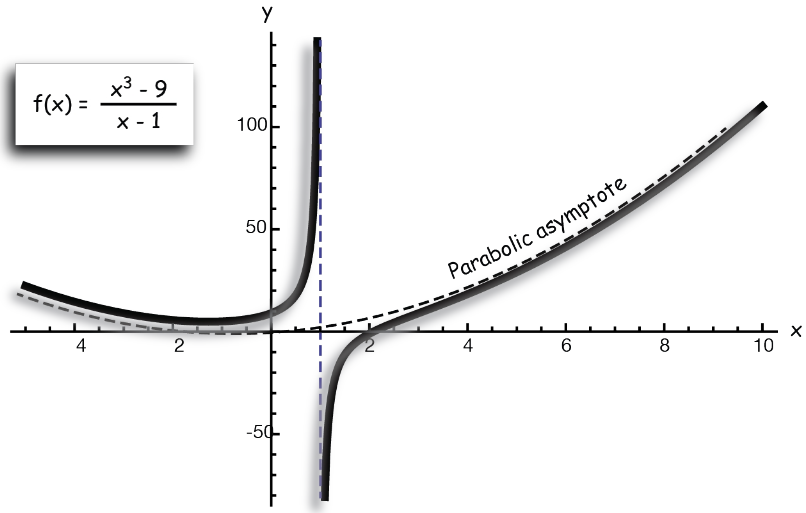
<u>Slant Asymptotes</u>

Rational functions can have non-horizontal or vertical asymptotes, too. The graph of the function at right shows a slant asymptote. A slant asymptote will be present when the degree of the polynomial in the numerator is one greater than the degree of the denominator. To find the equation of the slant asymptote, divide the numerator by the denominator and discard any remainder. The quotient, the equation of a line, is the equation of the slant asymptote.



Curved Asymptotes

When the degree of the numerator of a rational function differs by more than one from the degree of the denominator, the function will have curved asymptotes. One of the asymptotes of the function shown at right is the parabola $f(x) = x^2 - 1$.



Horizontal Asymptotes

Horizontal asymptotes are a little more difficult for us to visualize. Understanding horizontal asymptotes means being able to look at a function "dynamically". That is, we need to be able to look at the behavior of a function as the independent variable changes—especially as it grows to very large size in both the negative and positive directions. In mathematics, this is know as looking at the limits of a function $f(x)$ as $x \to \pm\infty$.

Take, for example, the parent function of all rational functions, $f(x) = 1/x$. As the denominator of any fraction grows, the value of that fraction gets smaller: ¼ is less than ⅓, and so on.... Now as x in the denominator of a function approaches ∞, the value of the function, for all practical purposes, is zero, although it never actually *gets* to zero. We say that "the limit of the function $f(x) = 1/x$ as $x \to \infty$ is zero."

That was pretty straightforward, but now what about a function like this:

$$f(x) = \frac{2x}{3x+1}$$

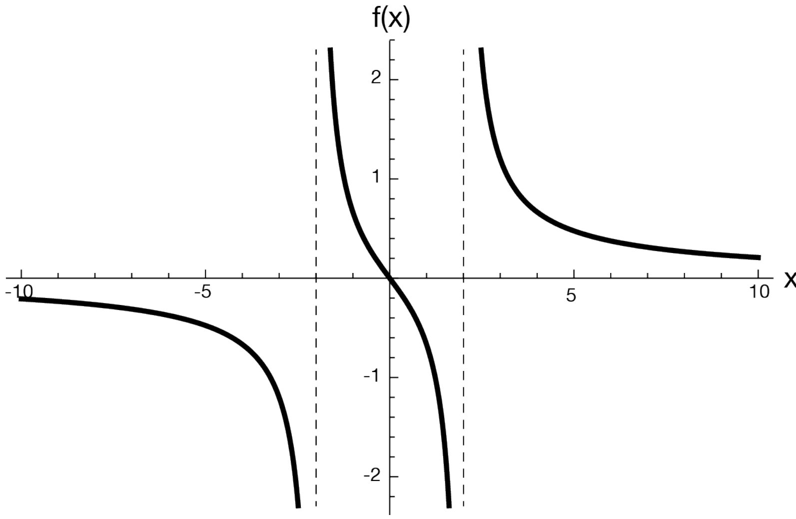
The way to analyze this function is to realize that first, for very large values of x (positive or negative), the +1 in the denominator won't make any difference at all, practically speaking. What's the difference between one trillion, and a trillion and one? That effectively reduces our problem to the limit as $x \to \infty$ of $\frac{2x}{3x}$, which is just $\frac{2}{3}$. $f(x)$ therefore has a horizontal asymptote at $y = ⅔$.

Here's another example: $f(x) = \frac{2x^2 - x - 7}{x^2 + x + 9}$.

Because the -7 and +9 are irrelevant when x becomes very large: $\lim_{x \to \infty} f(x) = \frac{2x^2 - x}{x^2 + x}$.

Now because the x^2 terms grow much faster than the x^1 terms as x gets larger, this limit can be further approximated by: $\lim_{x \to \infty} f(x) = \frac{2x^2}{x^2}$, which is just 2. So the limit of f(*x*) as $x \to \infty$ is just 2. The value of the function approaches 2, but never actually *gets* there.

Horizontal asymptotes are, different than vertical asymptotes in that they **can be crossed** by the function for certain values of the independent variable. Here's an example:



The function $f(x) = \frac{2x}{x^2 - 4}$.

Note that for large positive and negative values of *x*, the function value approaches *y* = 0, but that it actually *crosses* that asymptote at *x*=0.

Horizontal asymptotes are defined only for very large or very small values of the independent variable

Finding horizontal asymptotes: Look for $\lim_{x \to \pm\infty} f(x)$

<u>Transformations of Rational Functions.</u>

All of the usual transformations of functions can be applied to rational functions:

f(*x*) → f(*x*) + *k*	vertical translation by *k* units (upward if *k*>0)
f(*x*) → f(*x*-*h*)	horizontal translation by *h* units (right if *h*>0)
f(*x*) → Af(*x*)	vertical scaling by a factor of *A*
f(*x*) → $f\left(\frac{x}{C}\right)$	horizontal scaling by a factor of *C*
f(*x*) → -f(*x*)	reflection across the x-axis
f(*x*) → -f(*x*)	reflection across the y-axis

We can write a general rational function that includes all of these transformations like this:

$$f(x) = \frac{A}{C(x-h)} + k$$

Holes

Consider the function

$$f(x) = \frac{x^2+4x+4}{x+2}.$$

At first glance, we would note that its graph has a vertical asymptote at x = -2, but that's not right. Note that

$$f(x) = \frac{(x+2)(x+2)}{x+2},$$

so that one $(x+2)$ binomial divides into the other, leaving $(x+2)$ —which has no asymptotes—behind. Yet the function as originally written cannot take on the value x = -2. In this case, we say that x = -2 is a **hole** in the graph—a single point at which the function can have no value. We denote a hole by sketching a circle on the graph of $f(x)$ at $x=-2$.

Horizontal asymptotes: Some things to look for

1. If the degrees of the numerator and denominator are the same, the function will have a horizontal asymptote.
2. If the degree of the numerator is one larger than the degree of the numerator, the function will have a linear (slant) asymptote.
3. If the degree of the numerator is twice as large (or more) than the degree of the denominator, the function will have a curved asymptote. For example, if the degree of the numerator is 3 and that of the denominator is one, the function will have a parabolic asymptote.

Exercises

1. Sketch graphs of the following rational functions. Make sure to label any *x*- or *y*-intercepts, and asymptotes.

(a) $f(x) = \dfrac{x}{(x+2)(x-2)}$

(b) $f(x) = \dfrac{(2x+3)(x-1)}{(x+2)(x-1)}$

(c) $f(x) = \dfrac{x^3+2x^2-5x+2}{x^2-x-2}$

(d) $f(x) = \dfrac{(x^3+1)(x+1)}{x(x+1)}$

(e) $f(x) = \dfrac{x^2-9}{x^2-4}$

(f) $f(x) = \dfrac{(x^2-4)(x+3)}{x^2+2x-3}$